Journal of Sound and Vibration (1997) **204**(5), 735–753

A GENERALIZED TREATMENT OF THE ENERGETICS OF TRANSLATING CONTINUA, PART II: BEAMS AND FLUID CONVEYING PIPES

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(*Received* 16 *October* 1995, *and in final form* 30 *December* 1996)

The energetics of translating tensioned beams and fluid transporting pipes under fixed, simply supported and free boundary conditions are analyzed in a generalized manner. The conservative and non-conservative forces acting at the boundaries lead to energy transfer between the translating continua and the boundary supports. The forces and associated convective velocities are identified from the one-dimensional transport theorem. The group velocity and the wavenumbers of propagating and evanescent waves in the dispersive continua are defined. The time variation of total energy is represented in terms of the impedances and the reflection coefficients of the propagating waves in the continuum and the dynamic stability of the translating continua is discussed based on the energy expressions. The critical fluid speed in a cantilevered pipe, at which the resultant energy flux into the pipe at the free end vanishes, is determined by the use of travelling wave solutions.

1. INTRODUCTION

A translating, Euler–Bernoulli beam is the most common model of translating elastic systems such as magnetic tapes, band saws, power transmission chains and belts, textile and composite fibers, flexible manipulators and appendages under deploying motion, and pipes conveying fluid [1–6]. The translating beams and pipes may lose stability by either divergence or flutter. The translating continua undergo divergence instability when the centrifugal forces induced by moving particles exceed the restoring forces on the continua. The translating continuum with an asymmetric boundary configuration, such as a cantilevered pipe conveying fluid, can lose stability by flutter at high transport velocity.

The dynamic stability of elastic systems under conservative or non-conservative forces has received a significant attention [7–9]. Non-conservative systems can exist under static loads and conservative systems can occur under dynamic forces [10] in some cases. Contrary to the generally accepted notion that the total energy of free oscillation in conservative elastic systems is conserved, the total energy of a translating beam with fixed or simple supports varies periodically. The periodic variation in the total energy of the conservative gyroscopic system is due to energy flux resulting from boundary forces and the relative velocity of the moving continua at the stationary boundary. Energy flux into or out of a translating tensioned beam at a fixed or simple support was studied by Wickert and Mote [11]. They identified generalized forces leading to non-zero energy flux into the beam. Barakat [2] also showed that the total energy of a travelling Euler beam without tension is periodic.

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0022–460X/97/300735 + 19 $25.00/0/sv960946

As an extension of Part I [13] investigating the energetics of second order translating continua, this paper studies the energetics of fourth order continua, such as translating tensioned beams and fluid conveying pipes, with fixed, simply supported and free boundaries. The generalized forces and the corresponding velocities resulting in energy flux at the boundaries are identified. The group velocity and the wavenumbers of travelling and evanescent waves in the translating dispersive media are defined and used to quantify the energy transferred at the boundaries. Energy reflection coefficient, determining the energy transfer, are represented in terms of the impedances and the amplitudes of the propagating waves reflected from, and incident on, the boundary. The dynamic stability of the translating continua under both symmetric and asymmetric boundary configurations is then discussed based on the energy expression.

2. TRANSLATING TENSIONED BEAM

2.1. THE EQUATIONS OF MOTION

The linear equation of transverse motion of a tensioned Euler–Bernoulli beam travelling at constant speed v between two supports is [14, 15]

$$\rho\left(\frac{\partial^2 W}{\partial T^2} + 2V\frac{\partial^2 W}{\partial X\,\partial T} + V^2\frac{\partial^2 W}{\partial X^2}\right) - P_0\frac{\partial^2 W}{\partial X^2} + EI\frac{\partial^4 W}{\partial X^4} = 0, \qquad X \in (0, L), \tag{1}$$

where $W(X, T)$ is transverse displacement, ρ is linear density of the beam, P_0 is tension, and EI is flexural rigidity of the beam. Introduction of the dimensionless variables

$$x = X/L, \qquad w = W/L, \qquad t = (T/L^2)\sqrt{EI/\rho}, \qquad v = VL\sqrt{\rho/EI}, \qquad P = P_0(L^2/EI),$$

into equation (1) gives

$$w_{tt} + 2vw_{xt} + v^2 w_{xx} - Pw_{xx} + w_{xxxx} = 0, \quad x \in (0, 1). \tag{2}$$

Here the subscript notation indicates differentiation. The critical transport speed for divergence instability is determined from the time-independent equilibrium balance of equation (2), giving $v_c = \sqrt{P + \pi^2}$ for a simply supported beam, and $v_c = \sqrt{P + 4\pi^2}$ for fixed supports [16]. For sufficiently high tension, these critical speeds reduce to $v_c \simeq \sqrt{P}$, and the problem becomes essentially second order (string).

2.2. TRAVELLING WAVE CHARACTERISTICS

At subcritical speed, harmonic waves in a translating tensioned beam have the form

$$w(x, t) = A\,\mathrm{e}^{\mathrm{i}(\omega t - kx)}. \tag{3}$$

The wavenumber and frequency of the beam satisfy the dispersion relation

$$k^4 + (P - v^2)k^2 + 2v\omega k - \omega^2 = 0. \tag{4}$$

The wavenumber roots of equation (4) for the translating, infinite, tensioned beam include two real roots and a conjugate pair of roots with a positive (and usually small) real part. The positive and negative real wavenumbers describe downstream and upstream propagating waves and the conjugate root pair describes evanescent (non-propagating, spatially decaying) waves. The evanescent waves arise from a neighboring boundary, discontinuity or applied force [17, 18]. The four wavenumbers k_d, $-k_u$, and $k_R \pm \mathrm{i}k_I$ are shown in Figure 1 for $P = 0$ and 100. k_R vanishes at $v = 0$, and it is asymptotic to zero as tension increases and/or transport speed decreases. k_I is proportional to the time-rate of decay of the evanescent waves, and increases with tension. The wavenumber k_d for the downstream propagating wave decreases as v increases, and the upstream wavenumber k_u increases with increasing v. Non-vanishing v renders the downstream wavenumber smaller than the upstream one. The harmonic solution of equation (2) is

$$w(x, t) = A_d \, e^{-ik_d x} \, e^{i\omega t} + A_u \, e^{ik_u x} \, e^{i\omega t} + A_d^c \, e^{(-ik_R - k_I)x} \, e^{i\omega t} + A_u^c \, e^{(-ik_R + k_I)x} \, e^{i\omega t}. \tag{5}$$

A_d and A_u are the complex amplitudes of the downstream and upstream propagating waves, and A_d^c and A_u^c are the complex amplitudes of the downstream and upstream evanescent waves. The phase velocities of propagating waves are $c_d = \omega/k_d$ downstream and $c_u = \omega/k_u$ upstream. By the substitution of $\omega = kc$ into equation (4), the phase velocities are explicitly obtained:

$$c_d = v + \sqrt{P + k_d^2}, \qquad c_u = -v + \sqrt{P + k_u^2}. \tag{6}$$

2.3. ENERGY PROPAGATION: GROUP VELOCITY

The phase velocity c/k is the speed of propagation of geometrical features of the wave, while the group velocity $c_g = \partial\omega/\partial k$ measures the speed of propagation of the energy. The group velocities of downstream and upstream propagation waves in the translating tensioned beam become

$$c_{gd} = \frac{\partial \omega}{\partial k_d} = \frac{2k_d^3 + (P - v^2)k_d + v\omega}{\omega - vk_d}, \qquad c_{gu} = \frac{2k_u^3 + (P - v^2)k_u - v\omega}{\omega + vk_u}. \tag{7}$$

The group velocity of a stationary beam $c_g = 2k^3/\omega = 2\sqrt{\omega}$ is recovered from equation (7) when $P = v = 0$. The group and phase velocities in the translating beam are plotted in Figure 2(a–d) when $P = 0$ and 100. In a zero-tensioned beam, the energy of waves

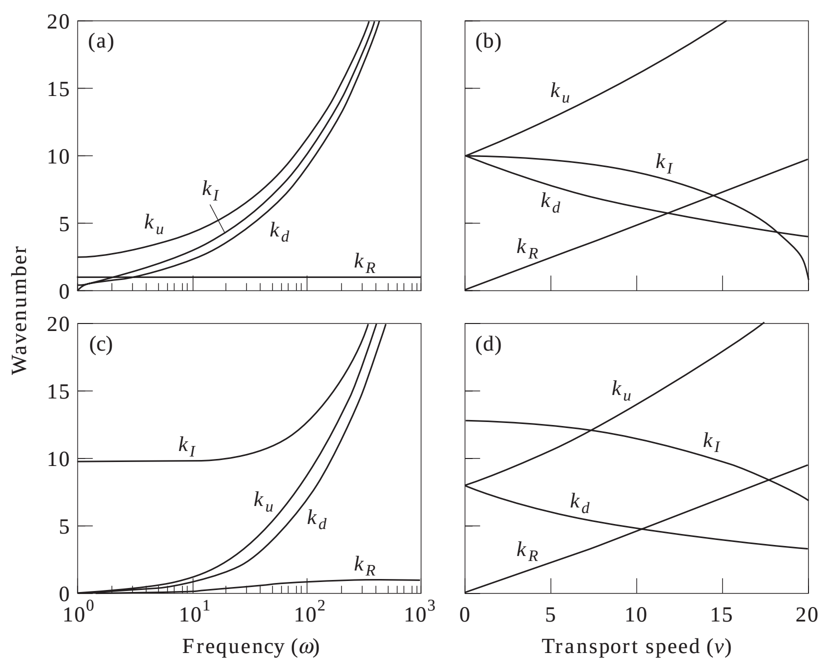

Figure 1. Wavenumbers of a translating tensioned beam: (a) $P = 0$ and $v = 2$; (b) $P = 0$ and $\omega = 100$; (c) $P = 100$ and $v = 2$; (d) $P = 100$ and $\omega = 100$.

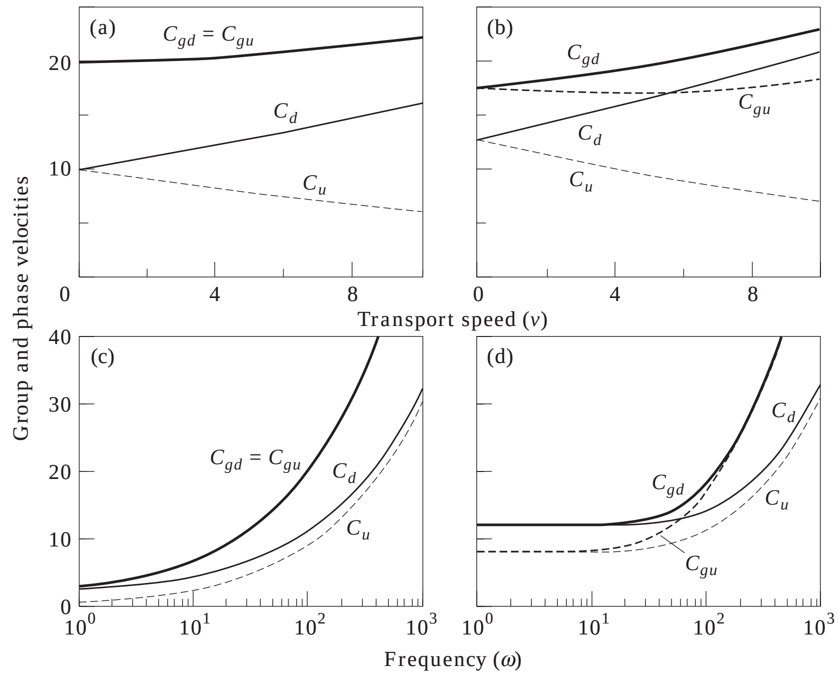

Figure 2. Group and phase velocities of downstream and upstream waves along a translating tensioned beam: (a) $P = 0$ and $\omega = 100$; (b) $P = 0$ and $\omega = 100$; (c) $P = 0$ and $v = 2$; (d) $P = 100$ and $v = 2$.

propagates both downstream and upstream at the same velocity $c_{gd} = c_{gu}$ for all v. The downstream and upstream group velocities in a tensioned beam are not the same for $v > 0$. At low frequency, the group velocity essentially equals the phase velocity (Figure 2(d)) as in a string model. As a group of waves propagates through the beam, $c_g > c$ indicates that high frequency waves appear at the front of the wave group, travel to the rear and disappear [17]. The difference between the group and phase velocities increases with frequency.

2.4. EXPLICIT WAVENUMBERS

The wavenumbers of the translating continua are explicitly obtained from equation (4) in special cases. When $v = 0$, the wavenumbers of a stationary tensioned beam are

$$k_d = k_u = \left(-P/2 + \sqrt{P^2/4 + \omega^2}\right)^{1/2}, \qquad k_R = 0, \qquad k_I = \left(P/2 + \sqrt{P^2/4 + \omega^2}\right)^{1/2}, \quad (8)$$

and when $P = 0$, the wavenumbers of a translating beam are

$$k_d = -\frac{v}{2} + \sqrt{\omega + \frac{v^2}{4}}, \qquad k_u = \frac{v}{2} + \sqrt{\omega + \frac{v^2}{4}}, \qquad k_R = \frac{v}{2}, \qquad k_I = \sqrt{\omega - \frac{v^2}{4}}. \quad (9)$$

When $v > 2\sqrt{\omega}$ and $P = 0$, k_I is imaginary. The critical transport speed, v_k, leading to $k_I = 0$ is plotted in Figure 3 when $\omega = 1$, 10 and 100. The critical speed v_k increases with tension and frequency. When v satisfies

$$v_k < v < v_c, \quad (10)$$

the two evanescent waves in equation (5) become propagating waves, and harmonic waves in the beam are represented by the following three downstream and one upstream propagation waves:

$$w(x, t) = A_{d1}\, e^{-ik_{d1}x}\, e^{i\omega t} + A_{d2}\, e^{-ik_{d2}x}\, e^{i\omega t} + A_{d3}\, e^{-ik_{d3}x}\, e^{i\omega t} + A_u\, e^{ik_u x}\, e^{i\omega t}, \tag{11}$$

where

$$k_{d1} = -\frac{v}{2} + \sqrt{\omega + \frac{v^2}{4}}, \quad k_{d2} = \frac{v}{2} + \sqrt{\frac{v^2}{4} - \omega}, \quad k_{d3} = \frac{v}{2} - \sqrt{\frac{v^2}{4} - \omega}, \quad k_u = \frac{v}{2} + \sqrt{\omega + \frac{v^2}{4}}. \tag{12}$$

When $v > v_k$, three downstream propagating waves with different phase velocities are permissible for a single harmonic wave. Therefore, the wave motion in a translating beam with a finite length, is characterized by the four propagating waves, when v is larger than the critical speed for a wave of fundamental frequency ($v > v_k(\omega_1)$). In this paper, it is assumed that $v < v_k(\omega_1)$ and the wave motion has the form (5).

3. PIPE CONVEYING FLUID

3.1. THE EQUATIONS OF MOTION

Consider a pipe (fourth order continuum) conveying fluids at a constant velocity U. If gravitational forces, internal damping, externally imposed tension and pressurization effects are neglected, the linear equation of transverse motion $W(X, T)$ of the pipe becomes [20, 22]

$$(m_f + m_p)\frac{\partial^2 W}{\partial T^2} + 2m_f U \frac{\partial^2 W}{\partial X\, \partial T} + m_f U^2 \frac{\partial^2 W}{\partial X^2} + EI\frac{\partial^4 W}{\partial X^4} = 0, \qquad x \in (0, L), \tag{13}$$

where m_f and m_p are mass densities of the fluid and pipe, and EI is the flexural rigidity of the pipe. For the variables

$$x = X/L, \qquad w = W/L, \qquad t = T/L^2\sqrt{EI/(m_f + m_p)}, \qquad u = UL\sqrt{m_f/EI},$$
$$\beta = m_f/(m_f + m_p),$$

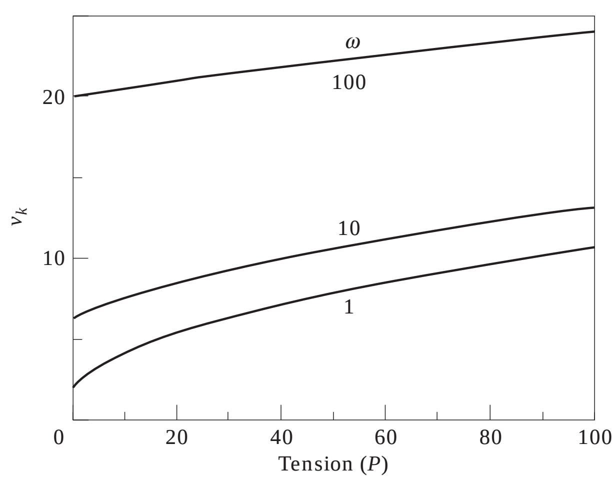

Figure 3. The speed v_k leading to $k_l = 0$ for $\omega = 1$, 10 and 100; $v_k = 2\sqrt{\omega}$ when $P = 0$.

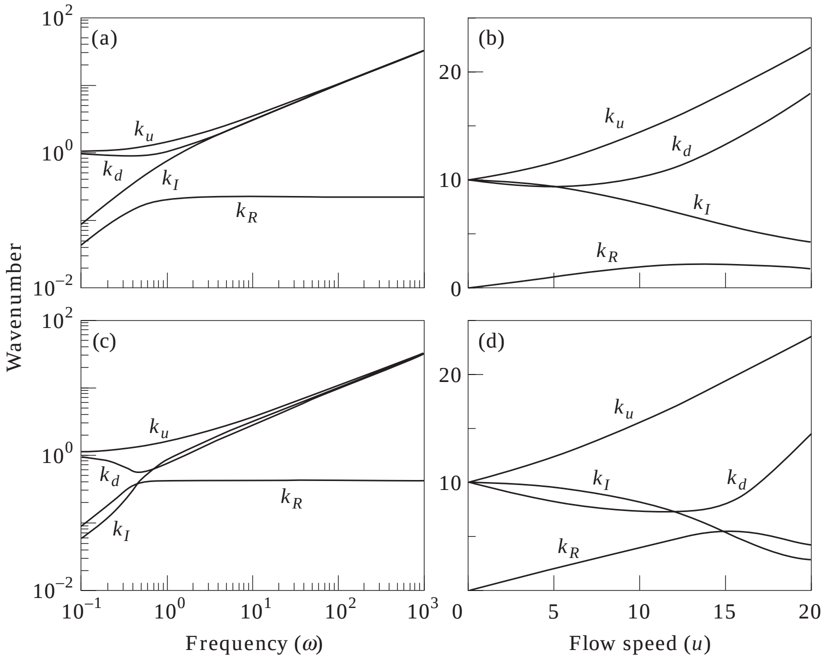

Figure 4. Wavenumbers of a pipe conveying fluid: (a) $\beta = 0{\cdot}2$ and $u = 1$; (b) $\beta = 0{\cdot}2$ and $\omega = 100$; (c) $\beta = 0{\cdot}7$ and $u = 1$; (d) $\beta = 0{\cdot}7$ and $\omega = 100$.

equation (13) is written in the dimensionless form

$$w_{tt} + 2\sqrt{\beta} u w_{xt} + u^2 w_{xx} + w_{xxxx} = 0, \qquad x \in (0, 1), \tag{14}$$

where $0 \leqslant \beta \leqslant 1$. When $m_f \gg m_p$, $\beta \to 1$ and equation (14) becomes identical to equation (2) for a translating beam with $P = 0$. The critical speeds for divergence instability are

$$u_c = \pi, \qquad u_c = 2\pi \tag{15}$$

for a pipe with simple and fixed supports, respectively. If tension is applied on the fluid-flowing pipe (14), the normalized linear equation becomes

$$w_{tt} + 2\sqrt{\beta} u w_{xt} + u^2 w_{xx} - P w_{xx} + w_{xxxx} = 0, \qquad x \in (0, 1), \tag{16}$$

where P is the pipe's tension. The translating tensioned beam (equation (2)) and the fluid conveying pipe (equation (14)) can be considered as special cases of equation (16).

3.2. TRAVELLING WAVE CHARACTERISTICS

For a harmonic travelling wave $w(x, t) = A\,\mathrm{e}^{\mathrm{i}(\omega t - kx)}$, the dispersion relation of equation (14) is

$$k^4 - u^2 k^2 + 2\sqrt{\beta} u \omega k - \omega^2 = 0. \tag{17}$$

The wavenumbers, k_d, $-k_u$, and $k_R \pm \mathrm{i}k_I$, which are roots of equation (17), are plotted in Figure 4, when $\beta = 0{\cdot}2$ ((a) and (b)) and $\beta = 0{\cdot}7$ ((c) and (d)). Contrary to the case of a translating tensioned beam, the wavenumber k_d of the downstream propagating wave decreases initially with frequency and increases after a particular frequency. The wavenumber k_d also decreases initially with flow speed and increases (or c_d decreases) at high flow speed. However the wavenumber k_u increases monotonically with increasing frequency and flow speed. The phase velocities of downstream and upstream travelling waves in the fluid conveying pipes are

$$c_d = \omega/k_d = \sqrt{\beta}u + \sqrt{k_d^2 - (1-\beta)u^2}, \qquad c_u = -\sqrt{\beta}u + \sqrt{k_u^2 - (1-\beta)u^2}. \tag{18}$$

4. ENERGETICS OF FOURTH ORDER CONTINUA

The total energy per unit length of the translating tensioned beam (2) is the sum of the kinetic and potential energy densities:

$$\hat{E} = \tfrac{1}{2}(w_t + vw_x)^2 + \tfrac{1}{2}(Pw_x^2 + w_{xx}^2). \tag{19}$$

The total mechanical energy $E(t)$ in the material particles within the fixed region $0 \leqslant x \leqslant 1$ is

$$E(t) = \int_0^1 \hat{E}\,\mathrm{d}x. \tag{20}$$

Because the beam transports mass at speed v, the time-rate of change of the total energy is expressed in the one-dimensional transport theorem [13]:

$$\dot{E}(t) = E_t + v\hat{E}|_0^1 + \mathscr{F}|_0^1, \tag{21}$$

where $(\dot{\ }) = \mathrm{d}/\mathrm{d}t$, $(\ )_t = \partial/\partial t$. E_t describes the local rate of change of energy within the domain, the second term of equation (21) represents the net rate of outward energy flux [19], and $\mathscr{F}$ denotes energy flux into the beam induced by the Coriolis and centrifugal forces at a free boundary. The non-conservative energy flux vanishes at fixed and simply supported boundaries. Substitution of equations (19) and (20) into equation (21) and use of equation (2) give

$$\dot{E}(t) = (Pw_x - w_{xxx})(w_t + vw_x)|_0^1 + w_{xx}(w_{xt} + vw_{xx})|_0^1 + \mathscr{F}|_0^1. \tag{22}$$

The first two terms on the right side of equation (21) are expressed by the rate of work by shear forces and bending moments at the boundaries. At an unconstrained boundary, the instantaneous transverse velocity is $w_t + vw_x$, and the material derivative of the slope is $w_{xt} + vw_{xx}$. Thus the energy fluxes, due to the boundary force and moment multiplied by the convective velocities are observed.

For the pipe transporting fluid (equation (14)), the energy density becomes

$$\hat{E} = \tfrac{1}{2}\beta(w_t + (u/\sqrt{\beta})w_x)^2 + \tfrac{1}{2}(1-\beta)w_t^2 + \tfrac{1}{2}w_{xx}^2, \tag{23}$$

and the total energy rate

$$\dot{E}(t) = -w_{xxx}\left(w_t + \frac{u}{\sqrt{\beta}}w_x\right)\Bigg|_0^1 + w_{xx}\left(w_{xt} + \frac{u}{\sqrt{\beta}} - w_{xx}\right)\Bigg|_0^1 + \mathscr{F}|_0^1$$

$$- \int_0^1 (1-\beta)w_{tt}\left(\frac{u}{\sqrt{\beta}}w_x\right)\mathrm{d}x \tag{24}$$

is obtained by substitution of equation (23) into equation (21) and use of equation (14). The non-conservative energy flux $\mathscr{F}$ occurs when flowing fluid exits into or out of unconstrained boundaries. The last term on the right represents energy flux from the inertial force under transverse motions within the domain. It always vanishes over a cycle of travelling waves because of the 90° phase difference between the local inertia force w_{tt} and the convective velocity $(u/\sqrt{\beta})w_x$ [13].

4.1. SIMPLE SUPPORT

For a tensioned travelling beam with simple supports ($w = w_{xx} = 0$ at $x = 0$ and 1), the temporal energy variation (22) wtih $\mathscr{F} = 0$ becomes [11]

$$\dot{E}(t) = (Pw_x - w_{xxx})vw_x|_0^1. \tag{25}$$

The shear force $Pw_x - w_{xxx}$ at the fixed end does work on the material with instantaneous transverse velocity vw_x. The sign of $(Pw_x - w_{xxx})vw_x$ in equation (25) is calculated using travelling waves. For a downstream wave $w(x, t) = A\,\mathrm{e}^{\mathrm{i}(\omega t - k_d x)}$ incident on $x = 1$, the force and the convective velocity at the end are

$$Pw_x - w_{xxx} = -\mathrm{i}(Pk + k_d^3)A\,\mathrm{e}^{\mathrm{i}(\omega t - k_d)}, \qquad vw_x = -\mathrm{i}vk_dA\,\mathrm{e}^{\mathrm{i}(\omega t - k_d)}. \tag{26}$$

Therefore, the force and convective velocity are in phase and the total energy increases at the downstream support. For an upstream wave, the force and transverse velocity vw_x are always 180° out of phase and energy flux is out of the continuum (negative) at the upstream end.

For the pipe conveying fluid between two simple supports, the energy variation becomes

$$\dot{E}(t) = -(u/\sqrt{\beta})w_{xxx}w_x|_0^1. \tag{27}$$

In this case, the shear force of the pipe does work with the relative transverse velocity between the flowing fluid and the stationary boundary.

4.2. FIXED SUPPORT

In a translating fixed-fixed beam, the total energy flux with $\mathscr{F} = 0$ becomes

$$\dot{E}(t) = -vw_{xx}^2(0, t) + vw_{xx}^2(1, t). \tag{28}$$

The work is done by the bending moment w_{xx} with the material derivative of the slope vw_{xx} on the translating beam. Energy flux into the beam is always positive at the downstream end and negative at the upstream end. Under a symmetric boundary configuration, the two energy fluxes with different signs at the downstream and upstream boundaries result in a periodic variation of the total energy of free motion. For a pipe conveying fluid with fixed boundaries, the total energy variation $\dot{E}(t)$ is obtained by replacing v by $u/\sqrt{\beta}$ in (28).

4.3. FREE BOUNDARY OF A PIPE CONVEYING FLUID

When the exit end of a pipe conveying fluid is unconstrained, the non-conservative force convected by discharging flow is the only source of energy input, because the shear force and bending moment at the end are zero. The non-conservative force acting on the downstream free end is determined by Hamilton's principle [24]:

$$\delta \int_{t_1}^{t_2} \mathscr{L}\,\mathrm{d}t + \int_{t_1}^{t_2} \delta W\,\mathrm{d}t = 0, \tag{29}$$

where $\mathscr{L} = T - V$ is the Lagrangian function made up of the kinetic energy (T) and potential energy (V) of the fluid-transporting pipe and δW is the virtual work convected by the non-conservative force $F_1(t)$ which is not included in the Lagrangian. The substitution of the Langrangian,

$$\mathscr{L} = \frac{1}{2}\int_0^1 \left(w_t^2 + 2\frac{u}{\sqrt{\beta}} w_x w_t + u^2 w_x^2 - w_{xx}^2 \right) \mathrm{d}x,$$

and the virtual work by the unknown non-conservative force,

$$\delta W = F_1(t)\delta w(1, t),$$

into equation (29) gives the boundary term at $x = 1$:

$$\{w_{xxx} - w_{xx} + (u/\sqrt{\beta})w_t + u^2 w_x + F_1\}\delta w = 0. \tag{30}$$

With the boundary conditions $w_{xx} = 0$ and $w_{xxx} = 0$ at the free end, equation (30) gives the non-conservative force

$$F_1(t) = -\sqrt{\beta} u w_t(1, t) - u^2 w_x(1, t). \tag{31}$$

In this case, the Coriolis and centrifugal forces are applied due to fluid particles passing through the end, unlike the case of the second order continuum where only Coriolis force acts on a free boundary [13]. The dimensional value of the normalized force (31),

$$F_1(t) = -m_f U W_T(L, T) + m_f U^2 W_X(L, T), \tag{32}$$

is equal to the result shown in references [20, 24]. The energy flux associated with the force is

$$\mathscr{F} = F_1(t) w_t(1, t) = -\sqrt{\beta} u\{w_t^2(1, t) + (u/\sqrt{\beta}) w_x(1, t) w_t(1, t)\}. \tag{33}$$

Accordingly, the time-rate of change of total energy in the pipe with a fixed–free boundary configuration becomes

$$\dot{E}(t) = -(u/\sqrt{\beta}) w_{xx}^2(0, t) - \sqrt{\beta} u\{w_t^2(1, t) + (u/\sqrt{\beta}) w_x(1, t) w_t(1, t)\}. \tag{34}$$

For small fluid speed, $w_x(1, t)w_t(1, t) > 0$ and energy decreases at both boundaries ($\dot{E}(t) \leqslant 0$). As u increases, $w_x(1, t)$ and $w_t(1, t)$ have opposite signs and energy is transferred into the pipe at the downstream free end. The expression for a critical fluid velocity, where the energy flux at $x = 1$ vanishes, is now examined by considering travelling waves. When a travelling wave $w(x, t) = A_d\,\mathrm{e}^{\mathrm{i}(\omega t - k_d x)}$ is incident on the free end, the force

$$F_1(t) = \mathrm{i}\sqrt{\beta} u A_d(\omega - (u/\sqrt{\beta})k_d)\,\mathrm{e}^{\mathrm{i}(\omega t - k_d)} \tag{35}$$

vanishes when

$$u/\sqrt{\beta} = \omega/k_d = c_d. \tag{36}$$

By the substitution of (18) into (36), the energy critical speed is given by

$$u_e = \sqrt{\beta} c_d = k_d\sqrt{\beta/(1-\beta)}. \tag{37}$$

If $u > u_e$, the force $F_1(t)$ is in phase with the transverse velocity at the free end, $w_t(1, t) = \mathrm{i}\omega A_d\,\mathrm{e}^{\mathrm{i}(\omega t - k_d)}$, and energy is transferred into the pipe over a complete cycle of oscillation ($\mathscr{F} > 0$). The energy critical speed for incident waves of frequency $\omega = 10$ and 100 are plotted in Figures 5(a) and (b). There exist three regions β describing three different mechanisms of energy transfer at the free end. At small β, two critical speeds exist and the magnitude of the wave energy increases only when the flow speed is between the two critical speeds. At the second region of β with one critical speed u_e, the wave energy increases when $u > u_e$. As β increases above a critical value, which is $\beta \simeq 0{\cdot}952$ in Figure

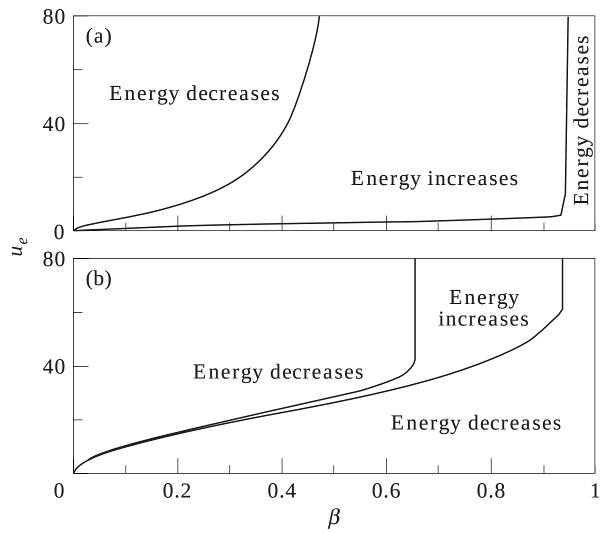

Figure 5. Energy critical speed u_e: (a) $\omega = 10$ and (b) $\omega = 1000$.

5(a) or $\beta \simeq 0{\cdot}935$ in Figure 5(b), no critical speed exists and then energy flux induced from the free end is always negative at any flow speed.

Over one cycle $T = 2\pi/\omega$ of free oscillation, the total energy, transferred from the flowing fluid into the cantilevered pipe at both downstream and upstream boundaries, becomes

$$\Delta W = \int_0^T \dot{E}(t)\,\mathrm{d}t = \int_0^T \left(-\frac{u}{\sqrt{\beta}} w_{xx}^2(0, t) - \sqrt{\beta} u \left\{ w_t^2(1, t) + \frac{u}{\sqrt{\beta}} w_x(1, t) w_t(1, t) \right\} \right) \mathrm{d}t. \quad (38)$$

The speed producing $\Delta W = 0$ is the critical speed for flutter instability of the cantilevered pipe and can be determined from equation (38). Because energy is always transferred out of the pipe at the upstream fixed end of the cantilevered pipe, the energy critical speed u_e in equation (37) gives the lower bound to the flutter speed u_f. When $u > u_f$, the total energy flux is positive and self-excited oscillation occurs [21, 23].

If the pipe has free inlet and fixed outlet boundaries, as would be obtained when sucking fluid through a cantilever, the non-conservative force

$$F_0(t) = \sqrt{\beta} u w_t(0, t) + u^2 w_x(0, t) \quad (39)$$

acts on the upstream free end. For an upstream incident wave $w(x, t) = A_u\,\mathrm{e}^{\mathrm{i}(\omega t + k_u x)}$, the force

$$F_0(t) = \mathrm{i}\sqrt{\beta} u A_u (\omega + (u/\sqrt{\beta}) k_u)\,\mathrm{e}^{\mathrm{i}\omega t} \quad (40)$$

is in phase with the boundary motion $w_t = \mathrm{i}\omega A_u\,\mathrm{e}^{\mathrm{i}\omega t}$, and the resultant energy flux is positive ($\mathscr{F} = F_0(t) w_t(0, t) > 0$). Because energy flux at the downstream fixed end is also positive ($(u/\sqrt{\beta}) w_{xx}^2(1, t) > 0$), the total energy over a cycle of free oscillation increases and the pipe loses stability at any flow speed. At different boundary conditions, energy fluxes are summarized in Table 1. It is noted that the sign of energy flux changes from negative to positive when the downstream boundary condition changes from free to fixed or simple support. This explains an interesting phenomenon, so called, *instability by adding support* [21, 25]. For a cantilevered pipe conveying fluid at high speed ($u_c < u < u_e$) which is above

TABLE 1

Energy flux in a pipe conveying fluid

Boundary	fixed	simple	free
Generalized Force	$-w_{xxx}$	w_{xx}	$-\sqrt{\beta} u w_t - u^2 w_x$
Convective velocity	$(u/\sqrt{\beta})w_x$	$(u/\sqrt{\beta})w_{xx}$	w_t
Energy flux at downstream end	+	+	− (if $u < u_e$), + (if $u > u_e$)
Energy flux at upstream end	−	−	+

the speed for divergence instability, if the free end was slightly touched with a finger, the pipe promptly buckled.

For a tensioned pipe conveying fluid (equation (16)) with free outlet and fixed inlet boundaries, the energy critical speed is obtained in a similar manner:

$$u_e = \sqrt{\beta/(1-\beta)(P + k_d^2)}. \tag{41}$$

As the special case ($\beta = 1$) of the tensioned pipe conveying fluid, the non-conservative force acting on a downstream free end of the translating tensioned beam (equation (2)) becomes

$$F_1(t) = -v w_t(1, t) - v^2 w_x(1, t). \tag{42}$$

In this special case, the energy critical speed (41) becomes infinite and energy flux into the system at the free end is always negative at any speed. The applied forces and energy fluxes at the different boundaries are summarized in Table 2.

5. GENERALIZED EXPRESSION FOR ENERGY TRANSFER

The energy contained in one wavelength $\lambda = 2\pi/k$ of a single harmonic wave $A\,\mathrm{e}^{\mathrm{i}(\omega t - kx)}$ propagating along a translating tensioned beam becomes

$$E_\lambda = \int_x^{x+\lambda} \hat{E}\,\mathrm{d}x = \pi k(P + k^2)AA^* = \pi\omega\mathscr{Z}AA^*, \tag{43}$$

where $\hat{E}$ is the energy density of the translating beam and

$$\mathscr{Z} = k(P + k^2)/\omega = (P + k^2)/c \tag{44}$$

is the mechanical impedance. The asterisk denotes the complex conjugate. By equation (6), the impedances of downstream and upstream waves are

$$\mathscr{Z}_d = (P + k_d^2)/\left(v + \sqrt{P + k_d^2}\right), \qquad \mathscr{Z}_u = (P + k_u^2)/\left(-v + \sqrt{P + k_u^2}\right). \tag{45}$$

TABLE 2

Energy flux in a translating tensioned beam

Boundary	fixed	simple	free
Generalized Force	$Pw_x - w_{xxx}$	w_{xx}	$-vw_t - v^2 w_x$
Convective velocity	vw_x	vw_{xx}	w_t
Energy flux at downstream end	+	+	−
Energy flux at upstream end	−	−	+

For a pipe conveying fluid, the impedances of downstream and upstream waves are

$$\mathscr{Z}_d = k_d^2/\left(\sqrt{\beta}u + \sqrt{k_d^2 - (1-\beta)u^2}\right), \qquad \mathscr{Z}_u = k_u^2/\left(-\sqrt{\beta}u + \sqrt{k_u^2 - (1-\beta)u^2}\right). \quad (46)$$

Consider a harmonic wave, propagating along the translating continuum, incident on a boundary of the medium. The energy ΔW, transferred into the continuum at the boundary by the wave reflection over a period equals the difference between the energies of the reflected and incident waves [13]:

$$\Delta W = E_r - E_i = \pi\omega(\mathscr{Z}_r A_r A_r^* - \mathscr{Z}_i A_i A_i^*). \quad (47)$$

Here $\mathscr{Z}_r$ and $\mathscr{Z}_i$ are the impedances of reflected and incident propagating waves and A_r and A_i are the complex amplitudes of the waves. The energy reflection coefficient,

$$R = E_r/E_i = \mathscr{Z}_r/\mathscr{Z}_i rr^*, \quad (48)$$

gives the exact expression for energy transfer between the translating continuum and the boundary support, in terms of impedances and the reflection coefficient $r = A_r/A_i$. The reflection coefficient is evaluated using the boundary conditions of the support.

6. WAVE AND ENERGY REFLECTIONS

The energy transfer between a translating media and a boundary support, given by equation (48), is determined by considering two waves propagating in opposite directions. Wave motions with imaginary and complex wavenumbers never cause energy flow [26]. Only propagating waves with real wavenumbers propagate energy. Exceptionally, evanescent waves are associated with energy flow only through the interaction between two opposite evanescent waves [27, 28]. We will consider only a propagating wave incident on the support to determine energy variation at a boundary. Attention to the propagating wave is reasonable because evanescent waves arising at a boundary decay rapidly and are negligible in most cases [29].

6.1. WAVE REFLECTION AT A DOWNSTREAM BOUNDARY

Consider a wave propagating downstream and impinging on the downstream boundary. An upstream propagating wave and an evanescent wave are generated. From equation (5), the harmonic wave motion at the downstream boundary is

$$w(x,t) = A_d\,\mathrm{e}^{-\mathrm{i}k_d x}\,\mathrm{e}^{\mathrm{i}\omega t} + A_u\,\mathrm{e}^{\mathrm{i}k_u x}\,\mathrm{e}^{\mathrm{i}\omega t} + A_u^c\,\mathrm{e}^{(-\mathrm{i}k_R + k_I)}\,\mathrm{e}^{\mathrm{i}\omega t}. \quad (49)$$

Substitutions of equation (49) into the following boundary conditions give reflection coefficients $r = A_u/A_d$ and $r^c = A_u^c/A_d$:

simple support: $w(1,t) = 0$, $w_{xx}(1,t) = 0$,

$$r = \frac{A_u}{A_d} = -\frac{k_d^2 - k_R^2 + k_I^2 - \mathrm{i}2k_Rk_I}{k_u^2 - k_R^2 + k_I^2 - \mathrm{i}2k_Rk_I}, \qquad r^c = \frac{A_u^c}{A_d} = \frac{k_d^2 - k_u^2}{k_u^2 - k_R^2 + k_I^2 - \mathrm{i}2k_Rk_I}; \quad (50)$$

fixed support: $w(1,t) = 0$, $w_x(1,t) = 0$,

$$r = -\frac{k_I + \mathrm{i}(k_d - k_R)}{k_I - \mathrm{i}(k_u + k_R)}, \qquad r^c = \frac{\mathrm{i}(k_u + k_d)}{k_I - \mathrm{i}(k_u + k_R)}; \quad (51)$$

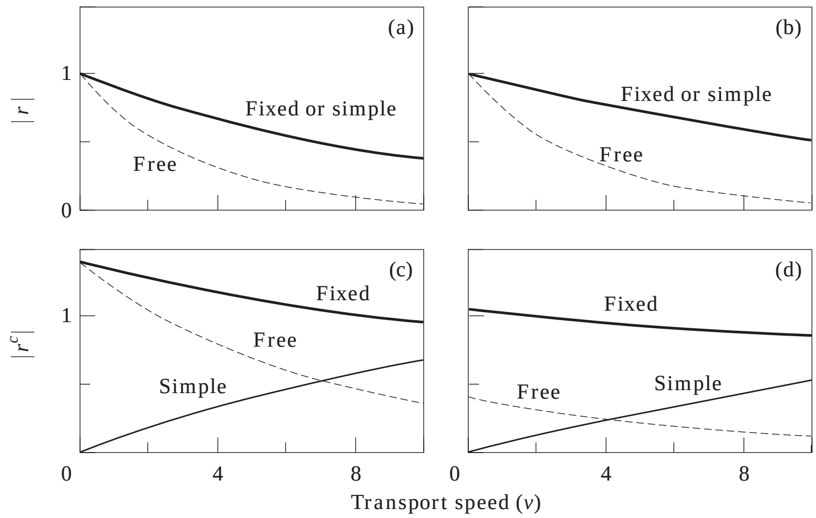

Figure 6. The reflection coefficients of a translating tensioned beam at downstream fixed, simple and free supports for a wave of $\omega = 100$: (a) r when $P = 0$; (b) r when $P = 100$; (c) r^c when $P = 0$; (d) r^c when $P = 100$.

free support: $w_{xx}(1, 0) = 0$, $w_{xxx}(1, t) = 0$,

$$r = -\left(\frac{k_d}{k_u}\right)^2 \frac{k_I + \mathrm{i}(k_d - k_R)}{k_I - \mathrm{i}(k_u + k_R)},$$

$$r^c = -\frac{\mathrm{i}k_d^2(k_u + k_d)}{k_I(k_I^2 - 3k_R^2 - 2k_u k_R) - \mathrm{i}\{k_I^2(k_u + 3k_R) - k_R^2(k_u + k_R)\}}. \tag{52}$$

The reflection coefficients r and r^c for a stationary beam are recovered by setting $v = 0$ and $P = 0$ ($k_d = k_u = k_I$ and $k_R = 0$). This leads to $r = -1$, $r^c = 0$ for a simply supported stationary beam, $r = -\mathrm{i}$, $r^c = -(1 - \mathrm{i})$ for a fixed support, and $r = -\mathrm{i}$, $r^c = (1 - \mathrm{i})$ for a free boundary. The magnitudes of the coefficients r and r^c are plotted in Figure 6 for the boundary supports (50)–(52) when $P = 0$ and 100. The magnitude of r decreases with transport speed for all the cases. Unlike the case of a translating string with $r = 1$ for any transport speed [13], the reflection coefficient r at a fixed or simple support of the translating beam varies with v. The magnitude of r at the fixed end is identical to that at the simple support, although the phases of the reflection coefficients are different. However, the magnitudes and phases of r^c at the fixed and simple supports are different. $|r^c|$ increases with the speed v at the simple support, while it decreases with v at the fixed end. As transport speed increases, $|r|$ and $|r^c|$ at a free boundary become smaller. Finally, the energy reflection coefficient using the reflection coefficient r of propagating waves is given by

$$R_d = \frac{E_u}{E_d} = \left(\frac{\mathscr{L}_u}{\mathscr{L}_d}\right) rr^* = \frac{k_u(P + k_u^2)}{k_d(P + k_d^2)} rr^*. \tag{53}$$

Similarly, the energy reflection coefficient for a fluid-transporting pipe satisfies

$$R_d = (k_u^3/k_d^3) rr^*. \tag{54}$$

6.2. WAVE REFLECTION AT AN UPSTREAM BOUNDARY

When a wave is incident upon the upstream boundary, it is reflected into a downstream propagating wave and a downstream decaying wave. The wave motion at $x = 0$ following the impinging upstream wave is

$$w(x, t) = A_d\, e^{-ik_d x}\, e^{i\omega t} + A_u\, e^{ik_u x}\, e^{i\omega t} + A_d^c\, e^{(-ik_R - k_I)x}\, e^{i\omega t}. \qquad (55)$$

Substitutions of equation (55) into simple, fixed, and free boundary conditions give the reflection coefficients at each support:

simple support: $w(0, t) = 0$, $w_{xx}(0, t) = 0$,

$$r = \frac{A_d}{A_u} = -\frac{k_u^2 - k_R^2 + k_I^2 + i2k_Rk_I}{k_d^2 - k_R^2 + k_I^2 + i2k_Rk_I}, \qquad r^c = \frac{A_d^c}{A_u} = \frac{k_u^2 - k_d^2}{k_d^2 - k_R^2 + k_I^2 + i2k_Rk_I}; \qquad (56)$$

fixed support: $w(0, t) = 0$, $w_x(0, t) = 0$,

$$r = -\frac{k_I + i(k_u + k_R)}{k_I - i(k_d - k_R)}, \qquad r^c = \frac{i(k_u + k_d)}{k_I - i(k_d - k_R)}; \qquad (57)$$

free support: $w_{xx}(0, t) = 0$, $w_{xxx}(0, t) = 0$,

$$r = -\left(\frac{k_u}{k_d}\right)^2 \frac{k_I + i(k_u - k_R)}{k_I - i(k_d - k_R)},$$

$$r^c = \frac{ik_u^2(k_u + k_d)}{k_I(k_I^2 - 3k_R^2 + 2k_dk_R) - i\{k_I^2(k_d + 3k_R) + k_R^2(k_d - k_R)\}}. \qquad (58)$$

The magnitudes of the reflection coefficients for the simple, fixed, and free upstream ends are plotted in Figure 7. Both $|r|$ and $|r^c|$ increase with transport speed v. From equation

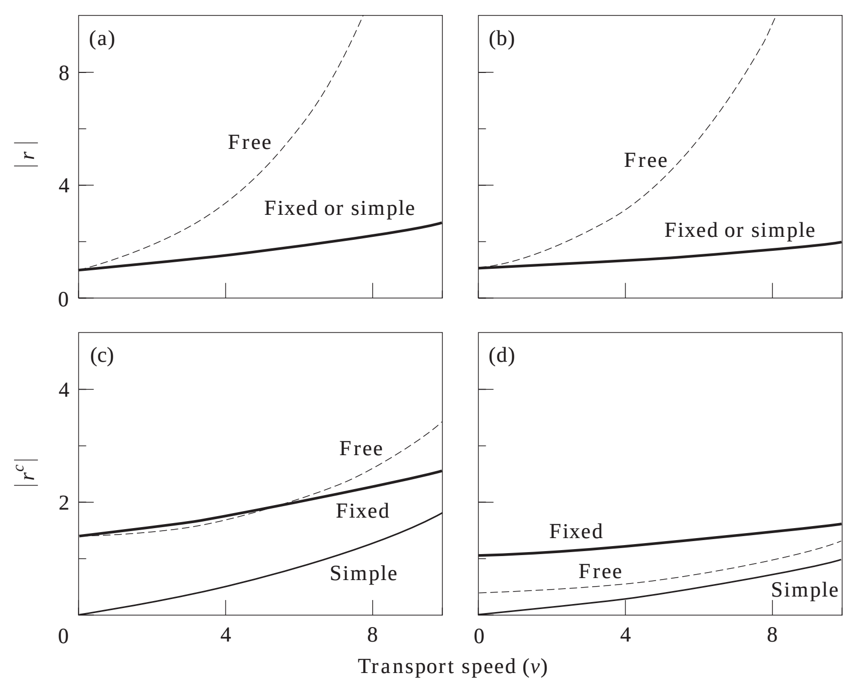

Figure 7. The reflection coefficients of a translating tensioned beam at upstream fixed, simple and free supports for a wave of $\omega = 100$: (a) r when $P = 0$; (b) r when $P = 100$; (c) r^c when $P = 0$; (d) r^c when $P = 100$.

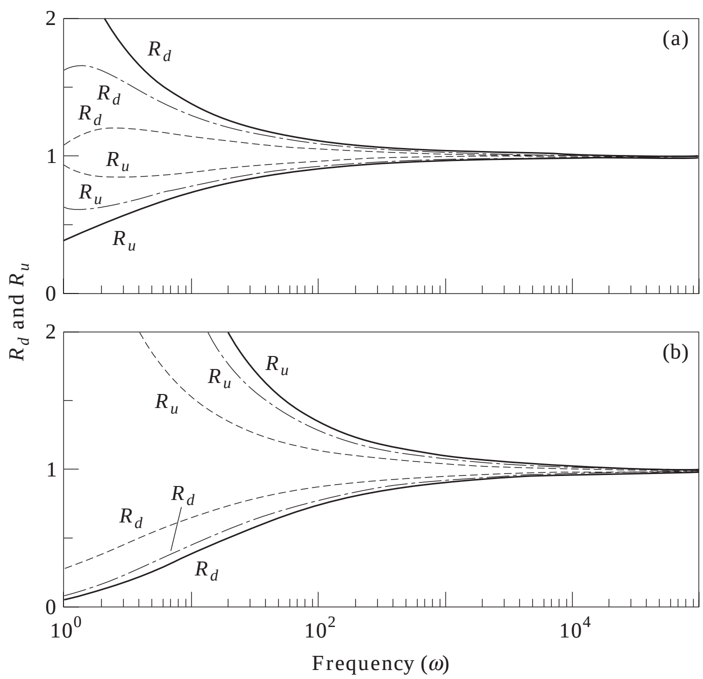

Figure 8. Energy reflection R_d and R_u of a pipe conveying fluid at $u = 1$: (a) simple or fixed support; (b) free boundary; - - - -, $\beta = 0{\cdot}2$; — · — · —, $\beta = 0{\cdot}7$; ——, $\beta = 1{\cdot}0$.

(48), the energy reflection coefficient R_u at the upstream boundary of a translating beam is

$$R_u = \frac{E_d}{E_u} = \frac{k_d(P + k_d^2)}{k_u(P + k_u^2)} rr^*. \tag{59}$$

Similarly, R_u for a fluid-transporting pipe in terms of wavenumbers and r becomes

$$R_u = (k_d^3/k_u^3) rr^*. \tag{60}$$

7. DISCUSSION

The energy reflection coefficients R_d and R_u of a pipe conveying fluid at $u = 1$ are plotted in Figures 8(a) and (b) for $\beta = 0{\cdot}2$, $0{\cdot}7$, and 1. As predicted in equations (25) and (28), energy flux induced by the flowing fluid is positive ($R_d > 1$) at a downstream fixed or simple support and negative ($R_u < 1$) at an upstream one. As β increases or frequency decreases, the magnitude of energy transferred at the downstream fixed (simple) support increases. At a free end of the fluid conveying pipe, the opposite energy transfer mechanism is shown in Figure 8(b). Energy is always transferred out of the pipe at the downstream free end. In this case, Coriolis and centrifugal forces are induced by mass transport through the free end and do negative work. When $\beta = 1$, the energy coefficients actually become those of a zero-tensioned beam translating at $v = 1$.

When a highly tensioned beam ($P = 100$) is axially moving, the dependence of R_d and R_u on wave frequency is illustrated in Figure 9. At low frequency, the high tension produces the dominant energy and the coefficients are essentially identical to those of a translating string in Part I [13]:

$$R_d = (\sqrt{P} + v)/(\sqrt{P} - v) = 1{\cdot}2222, \qquad R_u = (\sqrt{P} - v)/(\sqrt{P} + v) = 0{\cdot}8182. \tag{61}$$

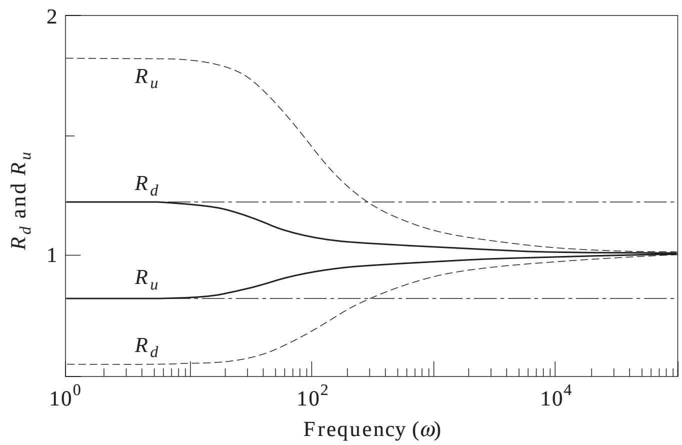

Figure 9. Energy reflection R_d and R_u of translating beams and strings with transport speed $v = 1$ and tension $P = 100$: (a) ——, translating beams (simple or fixed support); - - - -, fluid-flowing beam (free support), $\beta = 1$; — · — · —, translating string (fixed support).

As the limiting case ($\beta = 1$) of a tensioned pipe conveying fluid (equation (16)) with a free end, the translating tensioned beam gains energy at an upstream free boundary and loses energy at a downstream one. The energy change at each boundary is asymptotic to zero with increasing frequency because of the increasing contribution of bending to the total energy.

The energy reflection coefficients of a translating tensioned beam as a function v are plotted in Figure 10(a) when $\omega = 100$ and $P = 100$. Over one cycle of oscillation, the

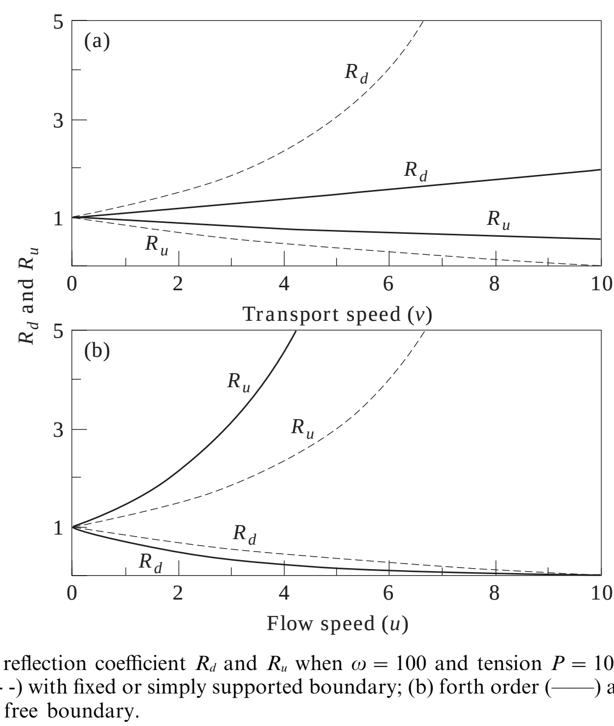

Figure 10. Energy reflection coefficient R_d and R_u when $\omega = 100$ and tension $P = 100$: (a) translating beam (——) and string (- - - -) with fixed or simply supported boundary; (b) forth order (——) and second order (- - - -) tensioned pipes with free boundary.

TABLE 3

Transferred energy at boundaries of a translating tensioned beam—sample problems

	$P = 100,\ v = 1,\ \omega = 1$				$P = 100,\ v = 1,\ \omega = 1000$			
k_d, k_u	0·0909, 0·1111				30·3470, 31·3458			
$k_R + ik_I$	0·0101 + i9·9504				0·4994 + i32·4191			
Boundary	Downstream		Upstream		Downstream		Upstream	
Impedance	$\mathscr{Z}_d = 9{\cdot}0913$		$\mathscr{Z}_u = 11{\cdot}1117$		$\mathscr{Z}_d = 30{\cdot}9825$		$\mathscr{Z}_u = 33{\cdot}9336$	
Group velocity	$c_{gd} = 11{\cdot}0012$		$c_{gu} = 9{\cdot}0019$		$c_{gd} = 61{\cdot}7747$		$c_{gu} = 61{\cdot}7652$	
Support	simple	fixed	simple	fixed	simple	fixed	simple	fixed
r	−1	−0·9998 − i0·0203	−1	−0·9998 − i0·0203	−0·9697 + i0·0005	−0·0487 − i0·9685	−1·0312 + i0·0005	−0·0518 − i1·0299
R	$R_d = 1{\cdot}2222$	$R_d = 1{\cdot}2222$	$R_u = 0{\cdot}8182$	$R_u = 0{\cdot}8182$	$R_d = 1{\cdot}0299$	$R_d = 1{\cdot}0299$	$R_u = 0{\cdot}971$	$R_u = 0{\cdot}971$

magnitude of energy transferred into or out of the fixed or simply supported beam is always smaller than that of a translating string with fixed supports, and the discrepancy increases with transport speed. In this case, the critical speed for buckling instability is $v_c = 10$ for the translating string, $v_c = 10{\cdot}482$ for the simply supported beam and $v_c = 11{\cdot}81$ for the fixed supported beam. When $\beta = 1$, the energy reflection coefficients of both the fourth and second order tensioned pipes at free boundaries are shown in Figure 10(b). Energy flux into or out of the fourth order pipe is larger than that at the second order one.

The wavenumbers, group velocities and reflection coefficients in the translating tensioned beam are calculated for two cases of low and high frequency in Table 3, when tension $P = 100$ and transport speed $v = 1$. For $\omega = 1$, the group velocities $c_g = 11{\cdot}0012$ downstream and $c_g = 9{\cdot}0019$ upstream are close to the phase velocities $c_d = \sqrt{P} + 1 = 11$ downstream and $c_u = \sqrt{P} - 1 = 9$ upstream. The group velocity downstream is always larger than that upstream because of translation of the elastic medium. The discrepancy is small at high frequency $\omega = 10^3$.

8. CONCLUSIONS

(1) At a fixed or simple support of the fourth order translating continua, the relative velocity between entering or exiting material particles and the stationary boundary leads to energy flux into the system. The energy flux is positive at a downstream boundary and negative at an upstream one.

(2) At a free boundary of a cantilevered pipe conveying fluid, the Coriolis and centrifugal forces which are induced by discharging fluids, do work on the boundary. The sign of the energy transfer depends on flow velocity. A flow speed leading to zero energy flux at the free end is determined by the incident wavenumber and the mass parameter β. It gives the lower bound for the flow speed at flutter of the cantilevered pipe.

(3) Energy transfer from the boundary support into the translating beam is quantified by the energy reflection coefficient without identifying explicitly the applied forces at the boundary. The coefficient depends on the mechanical impedance and the ratio of the reflected to incident wave amplitudes.

REFERENCES

1. J. A. Wickert and C. D. Mote, Jr 1988 *Shock and Vibration Digest* **20**, 3–13. Current research on the vibration and stability of axially moving materials.
2. B. Tabarrok, C. M. Leech and Y. I. Kim 1974 *Journal of the Franklin Institute* **297**, 201–220. On the dynamics of axially moving beams.
3. K. Tsuchiya 1983 *Journal of Guidance*, *Control and Dynamics* **6**, 100–103. Dynamics of a spacecraft during extension of flexible appendages.
4. A. E. Ibrahim and V. J. Modi 1986 *Acta Astronautica* **13**, 319–331. On the dynamics of beam type structural members during deployment.
5. J. Yuh and T. Young 1991 *American Society of Mechanical Engineers*, *Journal of Mechanical Systems*, *Measurement and Control* **113**, 34–40. Dynamic modeling of an axially moving beam in rotation: simulation and experiment.
6. M. P. Paidoussis and G. X. Li 1993 *Journal of Fluids and Structures* **7**, 137–204. Pipes conveying fluid: a model dynamical problem.
7. V. V. Bolotin 1963 *Nonconservative Problems of the Theory of Elastic Stability*. New York: Macmillan.
8. G. Herrmann 1967 *Applied Mechanics Reviews* **20**, 103–108. Stability of equilibrium of elastic systems subjected to nonconservative forces.
9. G. J. Simitses 1987 *Applied Mechanics Reviews* **40**, 1403–1408. Instability of dynamically-loaded structures.

10. H. Ziegler 1968 *Principles of Structural Stability*. Blaisdell Publishing Company.
11. J. A. Wickert and C. D. Mote, Jr 1989 *Journal of the Acoustical Society of America* **85**, 1365–1368. On the energetics of axially moving continua.
12. R. Barakat 1968 *Journal of the Acoustical Society of America* **43**, 533–539. Transverse vibrations of a moving thin rod.
13. S.-Y. Lee and C. D. Mote, Jr 1997 *Journal of Sound and Vibration*. A generalized treatment of the energetics of translating continua, part I: strings and tensioned pipes.
14. T. Chubachi 1958 *Bulletin Japan Society of Mechanical Engineers* **1**, 24–29. Lateral vibration of axially moving wire or belt form materials.
15. C. D. Mote, Jr 1965 *Journal of the Franklin Institute* **279**, 430–444. A study of band saw vibrations.
16. J. A. Wickert and C. D. Mote, Jr 1990 *American Society of Mechanical Engineers*, *Journal of Applied Mechanics* **57**, 738–744. Classical vibration analysis of axially moving continua.
17. K. F. Graff 1975 *Wave Motion in Elastic Solids*. Oxford: Oxford University Press. (Also 1991, New York: Dover Publications.)
18. B. R. Mace 1984 *Journal of Sound and Vibration* **97**, 237–246. Wave reflection and transmission in beams.
19. J. D. Achenbach 1975 *Wave Propagation in Elastic Solids*, 30–36. North-Holland Publishing Company.
20. T. B. Benjamin 1961 *Proceedings of the Royal Society of London* **261A**, 457–486. Dynamics of a system of articulated pipes conveying fluids; I. theory.
21. T. B. Benjamin 1961 *Proceedings of the Royal Society of London* **261A**, 487–499. Dynamics of a system of articulated pipes conveying fluids; II. experiments.
22. R. W. Gregory and M. P. Paidoussis 1966 *Proceedings of the Royal Society of London* **293A**, 512–527. Unstable oscillation of tabular cantilevers conveying fluid, Part I: theory.
23. R. W. Gregory and M. P. Paidoussis 1966 *Proceedings of the Royal Society of London* **293A**, 528–544. Unstable oscillation of tabular cantilevers conveying fluid, Part II: experiment.
24. D. B. McIver 1973 *Journal of Engineering Mathematics* **7**, 249–261. Hamilton's principle for systems of changing mass.
25. Y. Sugiyama, Y. Tanaka, T. Kishi and H. Kawagoe 1985 *Journal of Sound and Vibration* **100**, 257–270. Effect of a spring support on the stability of pipes conveying fluid.
26. D. J. Mead 1973 *Journal of Sound and Vibration* **27**, 235–260. A general theory of harmonic wave propagation in linear periodic systems with multiple coupling.
27. Y. I. Bobrovnitskii 1992 *Journal of Sound and Vibration* **152**, 175–176. On the energy flow in evanescent waves.
28. D. J. Mead 1994 *Journal of Sound and Vibration* **171**, 695–702. Waves and modes in finite beams: application of the phase-closure principle.
29. L. Cremer, M. Heckel and E. E. Ungar 1988 *Structure-Borne Sound*. Berlin: Springer-Verlag.